Instability of solitary wave solutions to long-wavelength transverse perturbations in the generalized Kadomtsev-Petviashvili equation with negative dispersion

Takeshi Kataoka and Michihisa Tsutahara
Graduate School of Science and Technology, Kobe University, Rokkodai, Nada, Kobe 657-8501, Japan
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The present paper is a more detailed article of the previously published Letter [Phys. Rev. Lett. **84**, 3065 (2000)], which discovered the transversely unstable solitary wave solutions of the generalized Kadomtsev-Petviashvili (GKP) equation with negative dispersion. In addition to detailed explanation of stability analysis to long-wavelength transverse perturbations, numerical calculation of the GKP equation is carried out here to study the transverse stability to perturbations of finite wavelength. The numerical results show that there is a short-wavelength cutoff to the transverse instability. Moreover, we reveal the existence of transversely unstable solitons.

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I. INTRODUCTION

The discovery of the transversely unstable solitary wave solutions of the generalized Kadomtsev-Petviashvili (GKP) equation with negative dispersion was published in Ref. [1]. Here we expand on that work and present some additional results: (i) there is a short-wavelength cutoff to this transverse instability and (ii) there exist transversely unstable solitons.

In the present study, only the negative dispersion case is treated. This case arises for describing any long gravity waves, such as internal waves and water waves with little surface tension effects. Therefore, this is the universal case we often encounter in describing long nonlinear dispersive waves in fluids.

Let us outline the preceding studies concerning the stability of solitary waves in the framework of the Korteweg–de Vries- (KdV-) type and KP-type evolution equations with negative dispersion. First, the one-dimensional stability of solitary wave solutions, or the stability to perturbations that depend only on the solitary wave's propagating direction was made by many authors [2–8] on the basis of the KdV-type equations of arbitrary nonlinear terms. It was found by these studies that the solitary wave solutions are one-dimensionally stable if the condition (8) shown below is satisfied.

The transverse stability of the one-dimensionally stable solitary wave solutions, or the stability to perturbations that depend on the transverse direction also has been made on the basis of the KP-type evolution equations. The first study was conducted by Kadomtsev and Petviashvili [9]. They studied the linear stability of solitary waves with respect to long-wavelength transverse perturbations in the framework of the classical KP equation whose nonlinear term is limited to the quadratic one. It was then found that the solitary waves are stable to such perturbations in a medium with negative dispersion. The same results were later reproduced by Kuznetsov *et al.* [10,11]. The complete linear stability analysis without restriction on the wavelength of perturbations was also conducted by many authors [12–15], and the results indicated stability again. Thus, the solitary waves are transversely stable in the framework of the classical KP equation.

The stability analysis in the framework of the GKP equation where its nonlinear term is generalized to arbitrary one, was first conducted by Bridges [16]. He made the linear stability analysis to transverse perturbations of long wavelength, or the small wavenumber. The solitary waves were then found to be at the neutral stability if only the leading-order effect of the small wavenumber is taken into account. Therefore the higher-order effects will determine the stability.

It was only a month later that the work of our group, which made an asymptotic analysis up to the next-order effect of the small wave number, was published [1]. Then existence of transversely unstable solitary waves was discovered. In the present paper, the rigorous analytical procedure and several numerical examples supporting this fact are presented. The outline of the present paper is as follows: After introducing the features of the solitary wave solutions in Sec. II, the stability analysis is given in Sec. III where the sufficient condition for the transverse instability is derived. In Sec. IV, we apply this criterion to some specific solitary wave solutions to show the existence of transversely unstable solitary waves. Some solitons are also included among them. In Sec. V, numerical results are presented. It is found that there is a short-wavelength cutoff to this transverse instability. In the last section, some concluding remarks are given.

II. SOLITARY WAVE SOLUTION

The GKP equation with negative dispersion is

$$\frac{\partial^2 A}{\partial t \partial x} + \frac{\partial^2}{\partial x^2}[f(A)] + \frac{\partial^4 A}{\partial x^4} + \frac{1}{2}\frac{\partial^2 A}{\partial y^2} = 0, \qquad (1)$$

where $f(A)$ is a given smooth function of A that satisfies $f(0)=0$ and $f'(0) \equiv df/dA|_{A=0}=0$. We seek a solution of Eq. (1) that is independent of y and approaches zero as $\xi \to \pm\infty$, where

$$\xi = x - vt, \tag{2}$$

and v is a positive real parameter. It is represented as

$$A = g(\xi). \tag{3}$$

Then g is governed by the following boundary-value problem:

$$\frac{d^2 g}{d\xi^2} + f(g) - vg = 0 \tag{4}$$

with the boundary condition

$$g(\xi) \to 0 \text{ as } \xi \to \pm\infty. \tag{5}$$

The necessary and sufficient condition for the existence of such nontrivial solution is that there exists a nonzero real constant g_0 and a positive real v satisfying

$$\int_0^{g_0} f(g)dg - \frac{v}{2}g_0^2 = 0 \tag{6}$$

[4]. For example, if $f(g)=g^{m/n}$ (m and n are relatively prime and n is odd; $g^{1/n}$ is defined to take real values), such g_0 and v always exist. If $f(g)=-g^{m/n}$, such g_0 and v exist only when m is even. When the above condition is satisfied, the boundary-value problem (4) and (5) possesses a nontrivial solution g that has the following two features: (i) $g(\xi)$ approaches zero exponentially as $\xi \to \pm\infty$; (ii) $g(\xi)$ is an even function of ξ, when the ξ coordinate is transferred appropriately such that its origin $\xi=0$ corresponds with the point where $|g|$ takes the maximum value. We call this solution g a solitary wave solution.

III. LINEAR STABILITY ANALYSIS TO LONG-WAVELENGTH TRANSVERSE PERTURBATIONS

Let us assume that the functional form of $f(g)$ is given and the condition (6) is satisfied. Without loss of generality, a solution of Eq. (1) can be expressed as

$$A = g(\xi) + \psi(\xi, y, t), \tag{7}$$

where $\psi(\xi, y, t)$ represents perturbations to the solitary wave solution g. In prior studies [4–8], the stability with respect to perturbations that have no dependence on y (one-dimensional stability) was investigated. According to their studies, the necessary and sufficient condition for the one-dimensional stability in the Lyapunov sense is to satisfy

$$\frac{dP}{dv} > 0, \tag{8}$$

where P is defined by

$$P = \frac{1}{2}\int_{-\infty}^{\infty} g^2 d\xi. \tag{9}$$

Here we investigate the transverse stability, or the stability with respect to perturbations that depend not only on the ξ direction but also on the y direction, of the one-dimensionally stable solitary wave solution. Since we make the linear stability analysis, ψ can be put in the following form:

$$\psi(\xi, y, t) = \phi(\xi)\exp(\lambda t + i\varepsilon y), \tag{10}$$

where ε is a given real constant and λ is a real or a complex constant which is determined by solving the equation for ϕ. Note that the solitary wave solution is transversely unstable if there exists a localized solution whose λ possesses the positive real part. Substituting Eq. (10) into Eq. (7) and then into Eq. (1) and omitting terms nonlinear with respect to ϕ, we obtain the following linearized equation for ϕ:

$$\lambda \frac{d\phi}{d\xi} - v\frac{d^2\phi}{d\xi^2} + \frac{d^2}{d\xi^2}[f'(g)\phi] + \frac{d^4\phi}{d\xi^4} - \varepsilon^2\frac{\phi}{2} = 0. \tag{11}$$

The boundary condition is

$$\phi(\xi) \to 0 \text{ as } \xi \to \pm\infty. \tag{12}$$

Equations (11) and (12) constitute the eigenvalue problem for ϕ whose eigenvalue is λ. We will solve this eigenvalue problem (11) and (12) whose g satisfies the condition (8). For the sake of analytical convenience, we restrict ourselves to the case of small ε, or the stability to long-wavelength transverse perturbations. Then, at the leading order, the last term on the left-hand side of Eq. (11) can be ignored, and the eigenvalue problem (11) and (12) possesses the following leading-order steady solution

$$\phi = \frac{dg}{d\xi}, \quad \lambda = 0. \tag{13}$$

This solution (13) will be subject to a slow time development if the last term on the left-hand side of Eq. (11), or the term of $O(\varepsilon^2)$ is recovered. In the followings, we investigate the asymptotic behavior for small ε under the assumption that its slow time development is described by the two different time scales of $O(\varepsilon^{-1})$ and $O(\varepsilon^{-2})$. So we express λ as

$$\lambda = \varepsilon\lambda_1 + \varepsilon^2\lambda_2. \tag{14}$$

The validity of this estimate is confirmed if the following analysis is consistent.

A. Core solution

First we look for a solution of Eqs. (11) and (12) whose appreciable variation occurs in ξ of the order of unity $[d\phi/d\xi = O(\phi)]$ (core solution), in a power series of ε:

$$\phi_C = \phi_{C0} + \varepsilon\phi_{C1} + \varepsilon^2\phi_{C2} + \cdots, \tag{15}$$

where

$$\phi_{C0} = \frac{dg}{d\xi}, \tag{16}$$

and the subscript C is attached to discriminate the type of solution (core solution).

Substituting the series (14) and (15) into Eq. (11) and arranging the same-order terms in ε, we obtain a series of equations for $\phi_{Cn}(n=1,2,\ldots)$:

$$\boldsymbol{L}_C\phi_{Cn} = G_n \quad (n = 1,2, \ldots), \tag{17}$$

where $\boldsymbol{L}_C$ is the linear operator defined by

$$\boldsymbol{L}_C = \frac{d^2}{d\xi^2} + f'(g) - v. \tag{18}$$

The inhomogeneous term G_n on the right-hand side of Eq. (17) consists of the lower-order solutions. Specifically,

$$G_n = \begin{cases} -\lambda_1 g + a_1 + b_1\xi & (n = 1), \\ \displaystyle\int_0^\xi \left(-\lambda_1\phi_{Cn-1} - \lambda_2\phi_{Cn-2} + \frac{1}{2}\int_0^\xi \phi_{Cn-2}d\xi \right) d\xi + a_n + b_n\xi & (n = 2,3, \ldots), \end{cases} \tag{19}$$

where a_n and b_n $(n=1,2,\ldots)$ are undetermined constants.

Now consider the linear inhomogeneous equation (17). Its homogeneous part, which is self-adjoint, has the nontrivial solution $dg/d\xi$ that approaches zero exponentially as $\xi \to \pm\infty$. Therefore, for the inhomogeneous equation (17) to have a solution that does not diverge exponentially as $\xi \to \pm\infty$, its inhomogeneous term G_n must satisfy the following relation (solvability condition):

$$\int_{-\infty}^{\infty} \frac{dg}{d\xi} G_n d\xi = 0, \tag{20}$$

since the left-hand side of Eq. (17) satisfies $\int_{-\infty}^{\infty}(dg/d\xi)\times(\boldsymbol{L}_C\phi_{Cn})d\xi=0$ for any ϕ_{Cn} that does not diverge exponentially as $\xi \to \pm\infty$. When the condition (20) is satisfied, the solution of Eq. (17) is expressed as

$$\phi_{Cn} = r_n\frac{dg}{d\xi} + \hat{\phi}_{Cn} \quad (n = 1,2, \ldots), \tag{21}$$

where r_n $(n=1,2,\ldots)$ is an arbitrary constant and $\hat{\phi}_{Cn}$ is the particular solution of Eq. (17).

At $n=1$, we easily find that, by putting $a_1=b_1=0$, the solvability condition (20) is identically satisfied and the solution of Eq. (17) at $n=1$ that satisfies the boundary condition (12) is obtained:

$$\phi_{C1} = r_1\frac{dg}{d\xi} - \lambda_1\frac{\partial g}{\partial v}, \tag{22}$$

where $\partial g/\partial v$ represents the derivative of g with respect to v for fixed ξ. It should be noted that, although the solution (22) itself does not satisfy the mass balance $\int_{-\infty}^{\infty}\phi d\xi=0$, we will see later [or after Eq. (48)] that this is automatically satisfied by taking into account a far-field solution.

At $n=2$, the solvability condition (20) is

$$\frac{dP}{dv}\lambda_1^2 + P = \left(\frac{g_0}{2} - b_2\right)M, \tag{23}$$

where $g_0=g(0)$. P and M are defined by Eq. (9) and

$$M = \int_{-\infty}^{\infty} g d\xi, \tag{24}$$

respectively. When the condition (23) is satisfied, we obtain the solution of Eq. (17) at $n=2$ that does not diverge exponentially as $\xi \to \pm\infty$. However, it never satisfies the boundary condition (12), since we easily find from Eq. (17) at $n=2$ that $\hat{\phi}_{C2}$ has the following values as $\xi \to \pm\infty$:

$$\phi_{C2}|_{\xi\to\infty} = -\frac{1}{v}\left[\frac{\lambda_1^2}{2}\frac{dM}{dv} + \frac{M}{4} + (r_1\lambda_1 + \lambda_2)g_0 + a_2 + \left(b_2 - \frac{g_0}{2}\right)\xi\right], \tag{25a}$$

$$\phi_{C2}|_{\xi\to-\infty} = -\frac{1}{v}\left[-\frac{\lambda_1^2}{2}\frac{dM}{dv} - \frac{M}{4} + (r_1\lambda_1 + \lambda_2)g_0 + a_2 + \left(b_2 - \frac{g_0}{2}\right)\xi\right]. \tag{25b}$$

For both quantities to be zero, which are the boundary condition (12), the terms proportional to ξ in Eqs. (25a) and (25b) must be zero first. Thus,

$$b_2 = \frac{g_0}{2}. \tag{26}$$

However, the terms which are independent of ξ in Eqs. (25a) and (25b) cannot be zero simultaneously, since the difference between $\phi_{C2}|_{\xi\to\infty}$ and $\phi_{C2}|_{\xi\to-\infty}$ [or $(\lambda_1^2 dM/dv+M/2)/v$] is generally not zero. This situation continues to the higher orders. The reason for such inappropriateness is that we did not take into account the balance between the first and the second terms on the left-hand side of Eq. (11). To achieve the balance, we need to introduce a shrunk coordinate with respect to ξ and seek a solution whose variation occurs slowly in ξ. This solution will be called a far-field solution. By accomplishing the connection with the core solution (15) and the far-field solution, the solution of Eq. (11) that satisfies the boundary condition (12) can be constructed. This procedure will be made from the following subsections. In this subsection, therefore, we concentrate on obtaining the core solution putting aside the boundary condition (12).

Now return to the main analysis. Substituting Eq. (26) into Eq. (23), we get

$$\lambda_1 = \pm i\sqrt{\frac{P}{dP/dv}}. \tag{27}$$

This criterion was obtained first by Kadomtsev and Petviashvili [9] in the framework of the classical KP equation and later by Kuznetsov *et al.* [10,11]. In the framework of the GKP equation, Bridges [16] derived this criterion. However, the real part of λ_1 is zero so that the stability of the solitary wave solution is not determined at this order. To know the stability, we must proceed to the next order.

At $n=3$, the solvability condition (20) is

$$\left\{-2\frac{dP}{dv}\lambda_2 + \frac{dM}{dv}[a_2 + (r_1\lambda_1 + \lambda_2)g_0]\right\}\lambda_1 + \left(\frac{r_1}{2}g_0 - b_3\right)M = 0, \tag{28}$$

which is obtained by the use of Eq. (17) at $n=1,2$ and integration by parts. This condition will generally give the nonzero real part of λ_2. Thus, the solution is obtained up to the orders that can determine the stability of the solitary wave solution. The specific values of a_2 and b_3, which are necessary for determining the value of λ_2, are given after accomplishing the connection between the core solution and the far-field solution. In the next subsection, the far-field solution is investigated.

B. Far-field solution

We seek a solution considering the balance between the first and second terms on the left-hand side of Eq. (11). To this end, in accordance with the introduction of $\varepsilon\lambda_1$ and $\varepsilon^2\lambda_2$ in Eq. (14), we here introduce two shrunk coordinates with respect to ξ:

$$X_1 = \varepsilon\xi, \quad X_2 = \varepsilon^2\xi. \tag{29}$$

We then look for the solution of Eq. (11) whose appreciable variation occurs in X_1 and X_2 of the order of unity $[\partial\phi/\partial X_1 = O(\phi),\ \partial\phi/\partial X_2 = O(\phi)]$, in a power series of ε:

$$\phi_F = \varepsilon^2\phi_{F2}(X_1,X_2) + \varepsilon^3\phi_{F3}(X_1,X_2) + \cdots, \tag{30}$$

where the subscript F is attached to discriminate the type of solution (far-field solution). The series of Eq. (30) starts from $O(\varepsilon^2)$ in accordance with the core solution not being able to satisfy the boundary condition as $\xi \to \pm\infty$ from this order [see the statement below Eq. (26)].

Substituting Eqs. (14), (29), and (30) into Eq. (11), and arranging the same-order terms in ε, a series of equations for ϕ_{Fn} $(n=2,3,\ldots)$ is obtained:

$$\boldsymbol{L}_F\phi_{F2} = 0, \tag{31}$$

$$\boldsymbol{L}_F\phi_{Fn} = H_n \quad (n = 3,4,\ldots), \tag{32}$$

where $\boldsymbol{L}_F$ is the linear operator defined by

$$\boldsymbol{L}_F = \lambda_1\frac{\partial}{\partial X_1} - v\frac{\partial^2}{\partial X_1^2} - \frac{1}{2}, \tag{33}$$

and H_n $(n=3,4,\ldots)$ is the inhomogeneous term given by

$$H_3 = \left(-\lambda_2\frac{\partial}{\partial X_1} - \lambda_1\frac{\partial}{\partial X_2} + 2v\frac{\partial^2}{\partial X_1\partial X_2}\right)\phi_{F2}, \tag{34}$$

$$\cdots.$$

The homogeneous equation (31) has a solution

$$\phi_{F2} = q_2\exp(kX_1) + \overline{q_2}\exp(\overline{k}X_1), \tag{35}$$

where q_2 and $\overline{q_2}$ are undetermined functions of X_2, and

$$k = \frac{\lambda_1}{2v}\left(1 + \sqrt{1 + \frac{2v}{|\lambda_1|^2}}\right), \quad \overline{k} = \frac{\lambda_1}{2v}\left(1 - \sqrt{1 + \frac{2v}{|\lambda_1|^2}}\right) \tag{36}$$

are both pure imaginary constants.

Equation (32) is a linear inhomogeneous equation and its homogeneous part has the nontrivial solutions $\exp(kX_1)$ and $\exp(\overline{k}X_1)$. For Eq. (32) to have a solution that does not diverge with respect to X_1, therefore, its inhomogeneous term must satisfy the following conditions:

$$\left|\int_0^{\pm\infty} H_n\exp(-kX_1)dX_1\right| < \infty,$$

$$\left|\int_0^{\pm\infty} H_n\exp(-\overline{k}X_1)dX_1\right| < \infty. \tag{37}$$

These conditions arise because the left-hand side of Eq. (32) satisfies $|\int_0^{\pm\infty}\exp(-kX_1)(\boldsymbol{L}_F\phi_{Fn})dX_1| < \infty$ and $|\int_0^{\pm\infty}\exp(-\overline{k}X_1)(\boldsymbol{L}_F\phi_{Fn})dX_1| < \infty$ for any bounded ϕ_{Fn}.

The conditions (37) at $n=3$ become

$$(2vk - \lambda_1)\frac{dq_2}{dX_2} - \lambda_2 k q_2 = 0, \quad (2v\overline{k} - \lambda_1)\frac{d\overline{q_2}}{dX_2} - \lambda_2\overline{k}\overline{q_2} = 0. \tag{38}$$

Solving these equations, we get

$$q_2 = c_{2+}\exp\left(\frac{\lambda_2 k}{2vk - \lambda_1}X_2\right),$$

$$\overline{q_2} = \overline{c_{2+}}\exp\left(\frac{\lambda_2\overline{k}}{2v\overline{k} - \lambda_1}X_2\right) \text{ for } X_1,X_2 > 0, \tag{39a}$$

$$q_2 = c_{2-}\exp\left(\frac{\lambda_2 k}{2vk - \lambda_1}X_2\right),$$

$$\overline{q_2} = \overline{c_{2-}}\exp\left(\frac{\lambda_2\overline{k}}{2v\overline{k} - \lambda_1}X_2\right) \text{ for } X_1,X_2 < 0, \tag{39b}$$

where c_{2+}, $\overline{c_{2+}}$, c_{2-}, and $\overline{c_{2-}}$ are undetermined constants. We can proceed to the higher orders in a similar way.

C. Connection of the core solution and the far-field solution

Here we carry out the connection of the core solution ϕ_C and the far-field solution ϕ_F. In the core region expressed by ϕ_C, the ordering of the far-field solution is reshuffled. That is, the far-field solution ϕ_{Fn} is expanded in the power series of X_1 (or $\varepsilon\xi$) and X_2 (or $\varepsilon^2\xi$):

$$\phi_{Fn} = (\phi_{Fn})_0 + \varepsilon\xi\left(\frac{\partial\phi_{Fn}}{\partial X_1}\right)_0 + \varepsilon^2\left[\frac{\xi^2}{2}\left(\frac{\partial^2\phi_{Fn}}{\partial X_1^2}\right)_0 + \xi\left(\frac{\partial\phi_{Fn}}{\partial X_2}\right)_0\right] + \cdots, \tag{40}$$

where the quantities in the parentheses with subscript 0, or $(\cdots)_0$, are evaluated at $X_1 = X_2 = 0$. With this reordering,

$$\phi_F = \varepsilon^2(\phi_{F2})_0 + \varepsilon^3(\phi_{F3})_0 + \varepsilon^4(\phi_{F4})_0 + \cdots + \varepsilon^3\xi\left(\frac{\partial\phi_{F2}}{\partial X_1}\right)_0 + \varepsilon^4\xi\left(\frac{\partial\phi_{F3}}{\partial X_1}\right)_0 + \cdots + \varepsilon^4\left[\frac{\xi^2}{2}\left(\frac{\partial^2\phi_{F2}}{\partial X_1^2}\right)_0 + \xi\left(\frac{\partial\phi_{F2}}{\partial X_2}\right)_0\right] + \cdots. \tag{41}$$

Collecting the same orders of ε, we obtain the reordered form (say, ϕ_{Fn}^*) of ϕ_{Fn}. For example, the new form (or ϕ_{F3}^*) of ϕ_F at the order of ε^3 is given by $(\phi_{F3})_0 + \xi(\partial\phi_{F2}/\partial X_1)_0$. After this reordering, we compare the forms of the two solutions ϕ_{Cn} and ϕ_{Fn}^* at each n and carry out their connection from $n=2$. The connection is accomplished if the conditions

$$\phi_{Cn} \sim \phi_{Fn}^* \text{ as } \xi \to \pm\infty, \tag{42}$$

are satisfied with differences being smaller than any inverse power of ξ.

At the order of ε^2, since $\phi_{F2}^* = (\phi_{F2})_0$, the connection is accomplished if

$$\phi_{C2} \sim (\phi_{F2})_{0+} \text{ as } \xi \to \infty, \tag{43a}$$

$$\phi_{C2} \sim (\phi_{F2})_{0-} \text{ as } \xi \to -\infty, \tag{43b}$$

where the quantities in the parentheses with subscript 0+ and 0− are evaluated as $X_1, X_2 \to 0+$ and $X_1, X_2 \to 0-$, respectively. Then, the connection conditions (43a) and (43b) are, from Eqs. (25a), (25b), (35), (39a), and (39b),

$$-\frac{1}{v}\left[\frac{\lambda_1^2}{2}\frac{dM}{dv} + \frac{M}{4} + (r_1\lambda_1 + \lambda_2)g_0 + a_2\right] = c_{2+} + \overline{c_{2+}}, \tag{44a}$$

$$-\frac{1}{v}\left[-\frac{\lambda_1^2}{2}\frac{dM}{dv} - \frac{M}{4} + (r_1\lambda_1 + \lambda_2)g_0 + a_2\right] = c_{2-} + \overline{c_{2-}}. \tag{44b}$$

At the order of ε^3, since $\phi_{F3}^* = (\phi_{F3})_0 + \xi(\partial\phi_{F2}/\partial X_1)_0$, the connection conditions are

$$\phi_{C3} \sim (\phi_{F3})_{0+} + \xi\left(\frac{\partial\phi_{F2}}{\partial X_1}\right)_{0+} \quad \text{as } \xi \to \infty, \tag{45a}$$

$$\phi_{C3} \sim (\phi_{F3})_{0-} + \xi\left(\frac{\partial\phi_{F2}}{\partial X_1}\right)_{0-} \quad \text{as } \xi \to -\infty. \tag{45b}$$

They consist of two different kinds of terms, i.e., those independent of ξ and those proportional to ξ. The connection conditions are obtained from each independently, that is, two conditions from the terms independent of ξ, and the other two from those proportional to ξ. The latter two relations contribute to the determination of the unknown constants at this order, and they are given by

$$-\frac{1}{v}\left\{\frac{\lambda_1}{v}\left[\frac{\lambda_1^2}{2}\frac{dM}{dv} + \frac{M}{4} + (r_1\lambda_1 + \lambda_2)g_0 + a_2\right] - \frac{r_1}{2}g_0 - \frac{\lambda_1}{4}\frac{dM}{dv} + b_3\right\} = kc_{2+} + \overline{k}\overline{c_{2+}}, \tag{46a}$$

$$-\frac{1}{v}\left\{\frac{\lambda_1}{v}\left[-\frac{\lambda_1^2}{2}\frac{dM}{dv} - \frac{M}{4} + (r_1\lambda_1 + \lambda_2)g_0 + a_2\right] - \frac{r_1}{2}g_0 + \frac{\lambda_1}{4}\frac{dM}{dv} + b_3\right\} = kc_{2-} + \overline{k}\overline{c_{2-}}. \tag{46b}$$

Moreover, from the boundary condition (12), we get, using Eq. (35) and the fact that the sign of the exponents in Eqs. (39a) and (39b) is the same as that of λ_2,

$$c_{2+} = \overline{c_{2+}} = 0 \text{ when } \lambda_2 > 0, \tag{47a}$$

$$c_{2-} = \overline{c_{2-}} = 0 \text{ when } \lambda_2 < 0. \tag{47b}$$

The six undetermined constants a_2, b_3, c_{2+}, $\overline{c_{2+}}$, c_{2-}, and $\overline{c_{2-}}$ are determined by the six equations (44a) and (44b); (46a) and (46b); and (47a) or (47b). Solving these equations, we get

$$a_2 = -\frac{\lambda_1^2}{2}\frac{dM}{dv} - \frac{M}{4} - (r_1\lambda_1 + \lambda_2)g_0, \quad b_3 = \frac{r_1}{2}g_0 + \frac{\lambda_1}{4}\frac{dM}{dv},$$

$$c_{2+} = \overline{c_{2+}} = 0,$$

$$c_{2-} = \left(1 + \frac{2v}{|\lambda_1|^2}\right)^{-1/2}\left[\frac{k}{\lambda_1}\left(\lambda_1^2\frac{dM}{dv} + \frac{M}{2}\right) - \frac{1}{2}\frac{dM}{dv}\right],$$

$$\overline{c_{2-}} = \left(1 + \frac{2v}{|\lambda_1|^2}\right)^{-1/2}\left[-\frac{\overline{k}}{\lambda_1}\left(\lambda_1^2\frac{dM}{dv} + \frac{M}{2}\right) + \frac{1}{2}\frac{dM}{dv}\right] \text{ when } \lambda_2 > 0,$$

$$a_2 = \frac{\lambda_1^2}{2}\frac{dM}{dv} + \frac{M}{4} - (r_1\lambda_1 + \lambda_2)g_0, \quad b_3 = \frac{r_1}{2}g_0 - \frac{\lambda_1}{4}\frac{dM}{dv},$$

$$c_{2+} = \left(1 + \frac{2v}{|\lambda_1|^2}\right)^{-1/2}\left[-\frac{k}{\lambda_1}\left(\lambda_1^2\frac{dM}{dv} + \frac{M}{2}\right) + \frac{1}{2}\frac{dM}{dv}\right],$$

$$\overline{c_{2+}} = \left(1 + \frac{2v}{|\lambda_1|^2}\right)^{-1/2}\left[\frac{\overline{k}}{\lambda_1}\left(\lambda_1^2\frac{dM}{dv} + \frac{M}{2}\right) - \frac{1}{2}\frac{dM}{dv}\right], \tag{48}$$

$$c_{2-} = \overline{c_{2-}} = 0 \text{ when } \lambda_2 < 0.$$

Now one can check that the obtained solution satisfies the mass balance up to the order of ε, or $\int_{-\infty}^{\infty}(\varepsilon\phi_{C1} + \varepsilon^2\phi_{F2})d\xi$

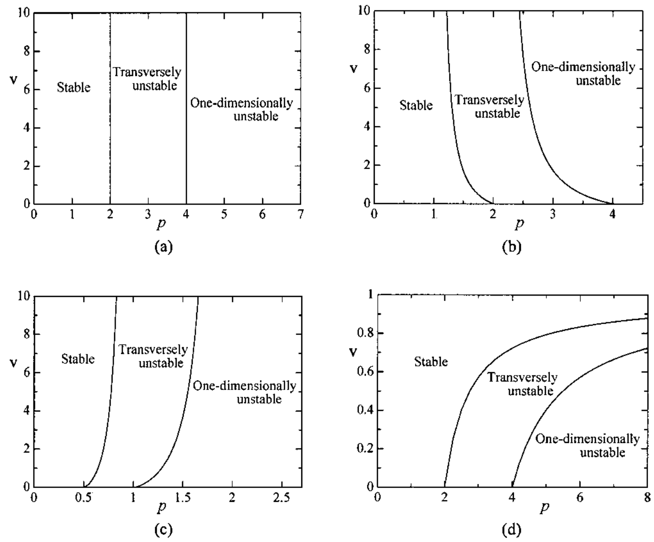

FIG. 1. Stability and instability regions on a parameter plane (p,v) of the solitary wave solutions given by (a) Eq. (54), (b) Eq. (57), (c) Eq. (58), and (d) Eq. (60).

$=0$ using Eqs. (22), (35), (39a), (39b), and (48).

Substituting the above results for a_2 and b_3 into the solvability condition (28), we obtain λ_2 as

$$\lambda_2 = \begin{cases} \pm \dfrac{PQ}{8(dP/dv)^2} & \text{if } Q < 0, \\ \text{no solution} & \text{if } Q > 0, \end{cases} \tag{49}$$

where

$$Q = \frac{d(M^2)}{dv}\frac{d}{dv}\left(\ln\left|\frac{P}{M}\right|\right). \tag{50}$$

Thus, when $Q<0$, we have a solution of the eigenvalue problem (11) and (12) whose λ has the positive real part (or $\lambda_2>0$). So the solitary wave solution is unstable to long-wavelength transverse perturbations when $Q<0$.

In contrast, when $Q>0$, there is no solution of the eigenvalue problem (11) and (12). In this case any perturbation modes are expected to be nonlocal. Since nonlocal perturbations cannot be excited by the solitary wave solution which is local, the solitary wave solution is stable to long-wavelength transverse perturbations when $Q>0$.

Now we can say that a sufficient condition for transverse instability is

$$Q < 0. \tag{51}$$

This criterion gives stability for the soliton of the classical KP equation, which is consistent with the results of the previous studies [12–15].

IV. TRANSVERSE INSTABILITY OF SPECIFIC SOLITARY WAVE SOLUTIONS

Let us apply the sufficient condition (51) for transverse instability to typical solitary wave solutions. Three different nonlinear functions $f(A)$ are considered. First,

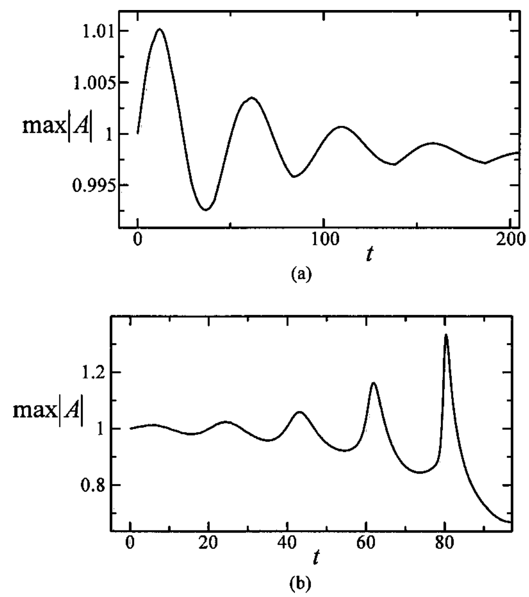

FIG. 2. Time dependence of the maximum amplitude $|A|$ on $y=0$ for the solution of the initial-value problem of the GKP equation (1) whose nonlinear function is given by Eq. (52) at (a) $p=1$, (b) $p=3$. The initial condition is Eq. (61) with $\varepsilon=\pi/20$ whose solitary wave solution g is given by Eq. (54) at (a) $p=1$ and $v=1$, (b) $p=3$ and $v=1$.

$$f(A) = \frac{p+2}{2}A^{p+1}, \tag{52}$$

where p represents

$$p = \frac{m}{n}. \tag{53}$$

Here m and n are relatively prime, n is odd, and $A^{1/n}$ is defined to take real values. The corresponding solitary wave solution is

$$g(\xi) = \left[\sqrt{v}\,\mathrm{sech}\left(\frac{p\sqrt{v}\xi}{2}\right)\right]^{2/p}, \tag{54}$$

where v is a positive real parameter. This solution is one-dimensionally stable when $p<4$ according to the one-dimensional stability criterion (8). The sufficient condition for the transverse instability is obtained by applying the instability criterion (51), which indicates the transverse instability when

$$2 < p < 4. \tag{55}$$

The stability region on a $p-v$ plane is illustrated in Fig. 1(a).

Second, consider

$$f(A) = (p+2)A^{p+1} + (p+1)A^{2p+1}, \tag{56}$$

where p is given by Eq. (53). There are two families of solitary wave solutions

$$g(\xi) = \left[\frac{v}{\sqrt{1+v}\cosh(p\sqrt{v}\xi)+1}\right]^{1/p}, \tag{57}$$

which exists for arbitrary values of m, and another solution

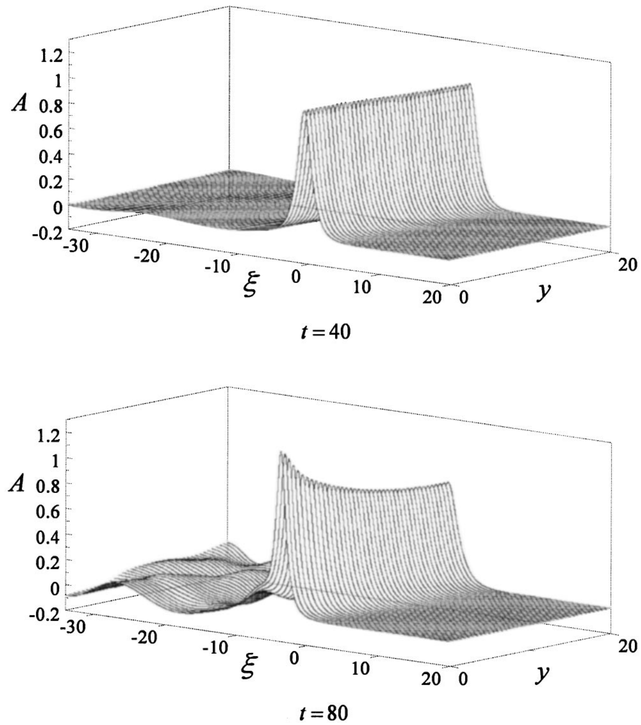

FIG. 3. The profiles of A at $t=40$ and 80 of the solution of the initial-value problem of the GKP equation (1) whose nonlinear function is given by Eq. (52) at $p=3$. The initial condition is Eq. (61) with $\varepsilon=\pi/20$ whose solitary wave solution g is given by Eq. (54) at $p=3$ and $v=1$.

$$g(\xi)=\left[\frac{v}{-\sqrt{1+v}\cosh(p\sqrt{v}\xi)+1}\right]^{1/p}, \tag{58}$$

which exists only when m is odd. By a trivial scaling transformation, these two solutions cover all the solitary wave solutions whose nonlinear function is given by any linear combination of A^{p+1} and A^{2p+1} with positive coefficient of A^{2p+1}. The stability of the solutions (57) and (58) depend not only on p but also on v. The stability regions on the parameter plane ($p-v$ plane) are shown in Figs. 1(b) and 1(c). One finds that the transversely unstable region adjoins the one-dimensional instability region on the side of the smaller p for both cases.

Third, consider

$$f(A)=(p+2)A^{p+1}-(p+1)A^{2p+1}. \tag{59}$$

The corresponding solitary wave solution is

$$g(\xi)=\left[\frac{v}{\sqrt{1-v}\cosh(p\sqrt{v}\xi)+1}\right]^{1/p}, \tag{60}$$

where v is less than 1 ($0<v<1$). By a trivial scaling transformation, this solution covers all the solitary wave solutions whose nonlinear function is given by any linear combination of A^{p+1} and A^{2p+1} with negative coefficient of A^{2p+1}. The stability region on the parameter plane ($p-v$ plane) is shown in Fig. 1(d). In this case also, the transversely unstable region exists on the side of the lower-order nonlinearities (or the side of the smaller p) of the one-dimensionally unstable region.

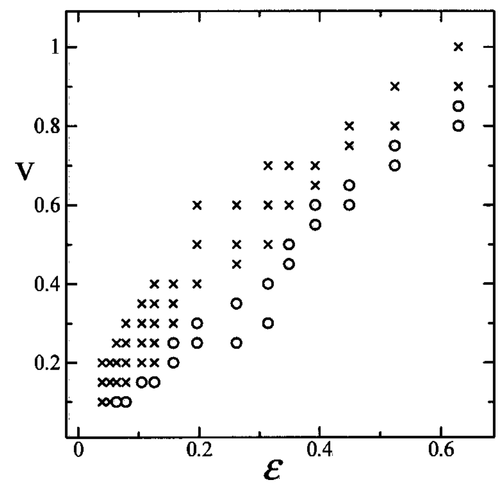

FIG. 4. Plots of stability (○) and instability (×) on a parameter plane (ε, v) of the solitary wave solution given by Eq. (54) at $p=3$ to transverse perturbations of wave number ε.

Finally, we arrange the results for $p=1$, whose solitary wave solutions are solitons. There are four kinds of solitons: Eqs. (54), (57), (58), and (60) at $p=1$. Their transverse stabilities are shown in Figs. 1(a)–1(d) at $p=1$. We see that the solitons (54), (57), and (60) are stable while the soliton (58) is transversely unstable. This is the first study that has revealed the existence of transversely unstable soliton propagating in a medium with negative dispersion. The nonlinear function of this unstable soliton is given by Eq. (56) at $p=1$. The corresponding soliton solutions are Eqs. (57) and (58) at $p=1$: one is stable and one is unstable.

V. NUMERICAL EXAMPLES

In this section we solve the GKP equation (1) numerically. The purpose is twofold. First, to show examples of transverse instability, and second, to investigate the transverse stability to perturbations of finite wavelength, or finite values of ε.

To these ends, we solve the GKP equation (1) under the following initial condition:

$$A(x,y,0)=g(x+0.1\cos\varepsilon y), \tag{61}$$

where ε is a given real constant. This initial condition (61) represents the solitary wave solution whose peak is distorted periodically with respect to y by the amplitude 0.1 and wave number ε. The usual finite-difference scheme is used for the numerical calculation. No disturbances ($A=0$) and the radiation condition [17] are applied far upstream and downstream, respectively, of the finite calculation region moving at speed v in the positive x direction, and the condition of symmetry is applied at $y=0$ and π/ε.

First, take the nonlinear function $f(A)$ given by Eq. (52) and the solitary wave solution g given by Eq. (54), respectively. Figures 2(a) and 2(b) show the time dependence of the maximum amplitude $|A|$ on $y=0$ when $\varepsilon=\pi/20$. We see that

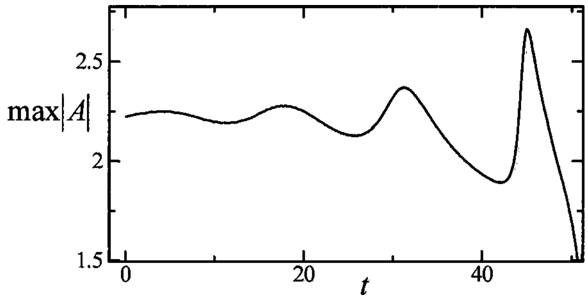

FIG. 5. Time dependence of the maximum amplitude $|A|$ on $y=0$ for the solution of the initial-value problem of the GKP equation (1) whose nonlinear function is given by Eq. (56) at $p=1$. The initial condition is Eq. (61) with $\varepsilon=\pi/20$ whose solitary wave solution (or the soliton solution) g is given by Eq. (58) at $p=1$ and $v=0.5$.

the amplitude grows as time elapses for the case of $p=3$. This is in accordance with the analytical result (55). Figure 3 shows the whole view of the wave (or the profile of A) for $p=3$, and the distortion of the solitary wave can be clearly seen.

The stability and instability, which are judged by the time dependence of the maximum amplitude $|A|$ on $y=0$, are shown in Fig. 4 for $p=3$ (the case of $p=1$ is always stable). According to the figure, the solitary wave is unstable as $\varepsilon \to 0$ (the left side of Fig. 4) independent of v, which agrees with the analytical result (55) for small ε. However, as ε increases, the solitary wave solution shows stable behavior from a certain critical value of ε, or $\varepsilon > \varepsilon_c(v)$. That is, there is a short-wavelength cutoff to the transverse instability.

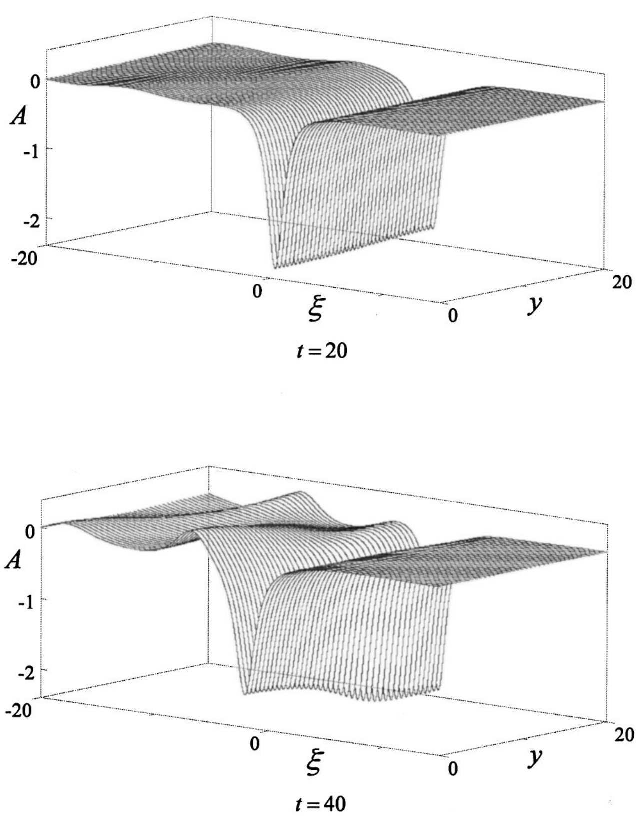

FIG. 6. The profiles of A at $t=20$ and 40 of the solution of the initial-value problem of the GKP equation (1) whose nonlinear function is given by Eq. (56) at $p=1$. The initial condition is Eq. (61) with $\varepsilon=\pi/20$ whose solitary wave solution (or the soliton solution) g is given by Eq. (58) at $p=1$ and $v=0.5$.

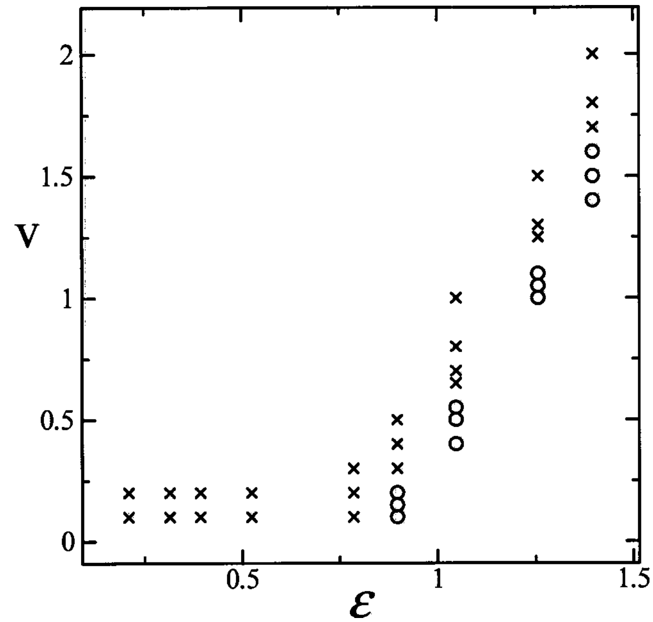

FIG. 7. Plots of stability (○) and instability (×) on a parameter plane (ε, v) of the soliton solution given by Eq. (58) at $p=1$ to transverse perturbations of wave number ε.

Next, take $f(A)$ given by Eq. (56) at $p=1$ and g given by the soliton solution (58) at $p=1$. Figure 5 shows the time dependence of the maximum amplitude $|A|$ on $y=0$ when $\varepsilon=\pi/20$. The figure indicates that the soliton is unstable, and agrees with the analytical result that the soliton solution (58) at $p=1$ is transversely unstable [see Fig. 1(c)]. Figure 6 shows the whole view of the wave (or the profile of A), and the distortion of the soliton can be clearly seen. The transverse stability for various values of ε is arranged in Fig. 7. In this case also, there is a short-wavelength cutoff to the transverse instability. This tendency is also confirmed in the other solitary wave solutions (57) and (60) that are transversely unstable for sufficiently small ε.

From the above numerical examples, existence of transversely unstable solitary waves including the soliton [or the solution (58) at $p=1$] has been strongly confirmed. We have also found that there is a short-wavelength cutoff to the transverse instability.

VI. CONCLUDING REMARKS

In the present study, the method of the stability analysis based on the GKP equation has been given fully in detail. For any solitary wave solutions investigated, there exists a region of transverse instability at the lower-order nonlinearities of the parameter plane than that of the one-dimensional instability. This result indicates the importance of checking the transverse stability of one-dimensionally stable solitary waves in fluids like in the water and stratified fluids.

Although the one-dimensional stability of solitary waves in fluids was investigated by several authors [18–20], the transverse stability was studied only by Bridges [21]. He treated the water waves. He derived the linear instability criterion to transverse perturbations of long wavelength, or the small wave number, and found the neutral stability at the leading order of the small wave number. To know the stability, however, the next order effect must be taken into account. According to our present study based on the GKP equation, the transverse instability appears at this next order. Therefore, it is very interesting to make linear stability analysis on the basis of the original governing equations of fluids, and investigate the transverse stability of one-dimensionally stable solitary waves in fluids. In fact, the authors made the analysis for the water wave recently and found the existence of transversely unstable solitary waves. The paper on this finding will be published soon [22].

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